AIAA JOURNAL
Vol. 31, No. 12, December 1993

Theoretical Aspects of Transition and Turbulence in Boundary Layers

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The main points of recent theoretical and computational studies on boundary-layer transition and turbulence are highlighted. The work is based on high Reynolds numbers and attention is drawn to nonlinear interactions, breakdowns, and scales. The article focuses in particular on truly nonlinear theories for which the mean-flow profile is completely altered from its original state. There appear to be three such theories, dealing with 1) nonlinear pressure-displacement interactions, 2) vortex/wave interactions, and 3) Euler-scale flows. Specific recent findings noted for these three, and in quantitative agreement with experiments, are the following. 1) Nonlinear finite-time breakups occur in unsteady interactive boundary layers, leading to sublayer eruption and vortex formation; here the theory agrees with experiments on the occurrence of the first spike. 2) Vortex/wave interactions give rise to finite-distance blowup of displacement thickness, then interaction and breakup (as "in 1"); this theory agrees with experiments on the formation of three-dimensional streets. 3) The Euler-scale and related theories lead to the prediction of turbulent boundary-layer microscale, displacement- and stress-sublayer thicknesses.

Nomenclature

A	= negative scaled displacement
b_1,b_2,b_3	= constants, during finite-time breakup (Sec. II)
c	= phase speed, during breakup
c.c.	= complex conjugate
F,G	= functions (Sec. IV)
m	= power, Eq. (30)
N	= power during breakup, Eq. (9)
P	= wave pressure in Sec. IV
$\bar{p},p$	= unscaled and scaled pressure
p_0,p_1	= pressure terms during breakup
R_δ	= Reynolds number, local
Re	= Reynolds number, global
t,T	= unscaled and scaled nondimensional time
$\bar{u},\bar{v},\bar{w}$	= unscaled nondimensional velocity components
u,v,w	= scaled nondimensional velocity components
u_0	= velocity profile at breakup
X,Y,Z	= scaled nondimensional rectangular coordinates
X_s,T_s	= position and time of breakup
x,y,z	= unscaled nondimensional rectangular coordinates
$\bar{x},\bar{y},\bar{z},\bar{t}$	= nondimensional coordinates scaled on the boundary-layer thickness
x_0	= typical station
x_1	= singular position
α	= scaled wave number
δ	= scaled boundary-layer displacement
$\bar{\lambda}$	= mean skin friction
λ	= skin-friction factor
ξ	= $\mathcal{O}(1)$ coordinate in breakup region
τ_w	= scaled skin friction
Ω	= scaled frequency

Subscript

n	= neutral

Presented as Paper 91-0331 at the AIAA 29th Aerospace Sciences Meeting, Reno, NV, Jan. 7–10, 1991; received April 9, 1992; revision received March 27, 1993; accepted for publication March 27, 1993.

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I. Introduction

THIS article is concerned with a number of central features of transition and turbulence in incompressible or compressible boundary layers. Stress is laid in particular on certain types of nonlinear interactions, e.g., viscous/inviscid, two-/three-dimensional, small-/large-scale, wave/vortex, on flow structures, and on the scales that are active. The nonlinear theoretical aspects involved are in an exciting state and in fact have a much wider application, relating to many flows of real interest including wakes, jets, free shear layers, rotating motions, oceanic circulations, and vortex flows.

Many kinds of boundary-layer transition are observed in practice, depending on the disturbance environment, e.g., with surface roughness or freestream unsteadiness. Two possible extremes are the following. First, there is the "fast," bypass, transition that is epitomized by pipe-flow transition and by the wake-passing effect, from an upstream row of rotor blades, on the boundary layers on stator blades in turbines. The other possible extreme of boundary-layer transition is "slow" transition, starting from linear Tollmien-Schlichting instability. This is exemplified by extensions of the classical experiments by Schubauer and Skramstad in the 1940s on tiny unsteady controlled disturbances introduced upstream into a basic planar laminar boundary layer. The typical progression here is linear two-dimensional decay, then growth, then nonlinear two-dimensional development followed by nonlinear three-dimensional motion. For sufficiently large (although still small) amplitudes, significant three-dimensional action can appear downstream with a preferred cross-stream wave number or a sustained vortex-like pattern of quite long streamwise length scale. Alternatively, there is the classic transition path of Klebanoff and Tidstrom from the 1950s and 1960s. There, nearly planar input disturbances upstream lead on within a relatively short distance to the formation of strongly three-dimensional streets downstream, in which turbulent bursts are initiated; later we describe theoretical-experimental comparisons for this path. In addition, intermittency may occur in any of the preceding stages experimentally, depending on the disturbance environment. Turbulent flow ensues some way downstream, but the nominally full turbulent state eventually reached may be regarded as broadly the same as that reached via the fast type of bypass transition described earlier.

The present article summarizes research in progress directed toward greater theoretical understanding of both the transition process and ultimately, of course, a systematic account of turbulent flow and turbulence modeling if possible. It is intended that this theoretical understanding, e.g., of scales, should help to guide direct computational simulations, as well as providing parameterization and comparisons with experiments. Here we attempt to highlight the central points of the three main nonlinear theories involved, i.e., 1) pressure-displacement interactive boundary-layer theory, 2) high-frequency-cum Euler-scale theory, and 3) vortex/wave interaction theory. This is as opposed to providing a broad review of what is a vast research area and would require enormous journal space. Theories 1–3 are described in turn in Secs. II–IV. We address not only the slower types of transition but also the bypass types, referred to later on. Moreover, attention is drawn to nonlinear interactions which completely alter the mean-flow profile, and this means the preceding three. The only major assumption in the theories is that the typical Reynolds number Re is large, which seems in line with the practical aerodynamic interest in enormous Re values.

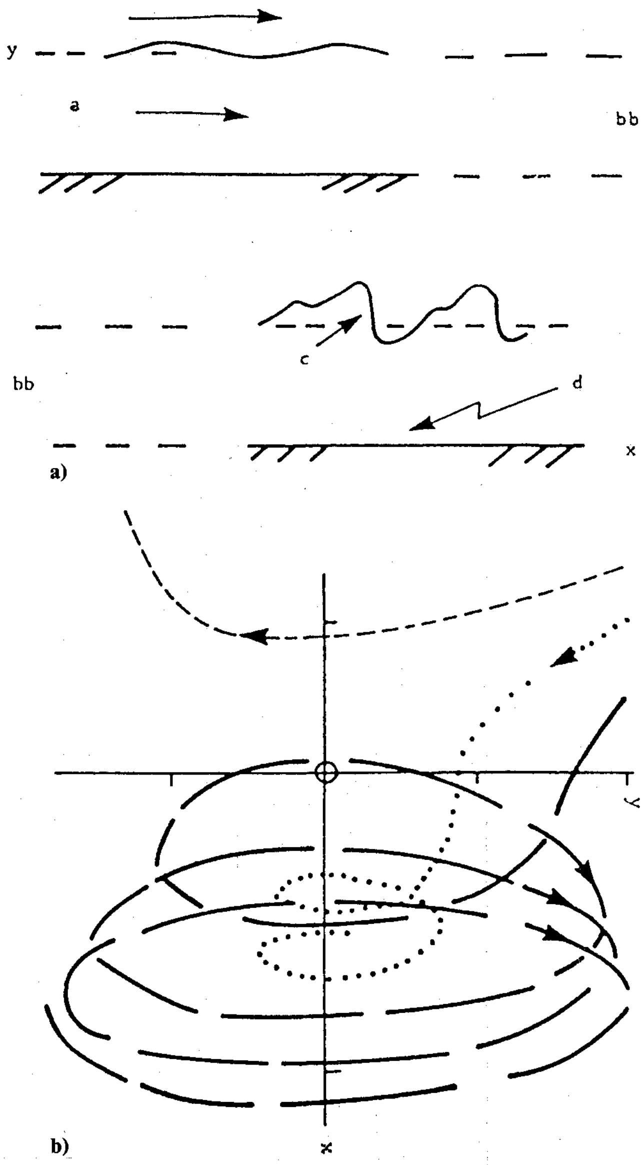

Fig. 1 Finite-time breakup[2] due to IBL transition, e.g, as a TS wave becomes strongly nonlinear, in the upper graph. Small input v,p' (at a) develops nonlinearly (via bb), to produce increasingly large output v,p' (at c) and large wall shear τ_w (at d) during the breakup. Table 1 summarizes the comparisons between computations[20] and theory[2] for the power response (B) in the wall-shear singularity, at various Reynolds numbers. The lower graph shows computed solutions[28] for the particle trajectories during subsequent vortex formation and wind-up.

Section II considers the first nonlinear theory 1, that of nonlinear Tollmien-Schlichting (TS) or interacting-boundary-layer (IBL) interactions, which are controlled by the unsteady IBL equations. Emphasis is given to the recent findings of nonlinear finite-time breakups,[1,2] detailed comparisons with computations, and the repercussions. This breakup refers to a singularity that is encountered in the IBL system in general within a finite scaled time and involves the scaled pressure gradient and skin friction, among other properties, becoming unbounded locally. A local change of scales is therefore induced. The repercussions of this are concerned principally with local sublayer eruption and vortex formation; see Fig. 1. Section III moves on to the second nonlinear theory 2, that of increased frequencies and Euler-scale flows.[3,4] The latter have all of their dominant length scales being comparable with the boundary-layer thickness and are therefore controlled by the unsteady nonlinear Euler system. There is also a viscous sublayer which can itself, by bursting, provide extra vorticity input into the main Euler region. The overall structure here yields predictions for turbulent boundary layers (see also Sec. V) which are in agreement with those observed experimentally or used in simple turbulence models. Section IV considers other three-dimensional features (Figs. 4 and 5). It is noted that, in particular, nonlinear interactions can occur sooner, i.e., at lower amplitude, in the three-dimensional version compared with the two-dimensional version. Special mention is made of the recent findings on vortex/wave nonlinear interactions,[5,6] which constitute the third nonlinear theory 3, and three-dimensional finite-time breakups. These theoretical nonlinear three-dimensional interactions also agree well qualitatively and sometimes quantitatively with the experimental findings just mentioned. Section V gives a brief further discussion, including tentative views on turbulent flow.[3,4]

The theory seems to be quite promising overall, although there is still a very long way to go in the three-dimensional context. The work in Secs. II–IV fits with observations and modeling[7,8] on turbulent boundary layers (see also Secs. III and V), which is encouraging, and there is significant quantitative agreement with experiment.[9] In particular, theoretical-experimental agreement is found in recent papers[10–12] concerning the Klebanoff and Tidstrom[13] and Nishioka et al.[14] measurements among others and the occurrence of the first spike in transition. Other recent theoretical work of this type is tackling certain nonlinear three-dimensional initial-value problems with regard to the evolution of nonlinear spot disturbances.[15,16]

II. Nonlinear Interacting Boundary-Layer Transitions and Breakup (Theory 1)

We consider two-dimensional nonlinear features first, as a guideline. Given that the Reynolds number Re (based on chord length and freestream speed) can be taken to be a large parameter, we address the Navier-Stokes equations but scale the nondimensional velocities $\bar{u}$ and $\bar{v}$ in the x and y directions, and the nondimensional pressure $\bar{p}$, in the Tollmien-Schlichting (TS) lower-branch fashion.[1–3,17] That is equivalent to the triple-deck scalings:

$$(\bar{u},\bar{v}) = (Re^{-1/8}\lambda^{1/4}u,\ Re^{-3/8}\lambda^{3/4}v) + \ \ldots \tag{1}$$

$$\bar{p} = Re^{-1/4}\lambda^{1/2}p(X,T) + \ \ldots \tag{2}$$

$$(x,y,t) = (x_0 + Re^{-3/8}\lambda^{-5/4}X,\ Re^{-5/8}\lambda^{-3/4}Y,\ Re^{-1/4}\lambda^{-3/2}T) \tag{3}$$

Here the scalings [Eqs. (1–3)], and those for $\bar{w}$ and z in the three-dimensional setting addressed later, apply in the so-called lower deck, which is a viscous sublayer close to the surface. The factor $\lambda = \lambda(x_0)$ is the reduced skin friction of the oncoming undisturbed $\mathcal{O}(Re^{-1/2})$ thick boundary layer, e.g., in Blasius flow $\lambda(x_0) \propto x_0^{-1/2}$, at the typical $\mathcal{O}(1)$ station

$x = x_0$. The flow problem in two dimensions comes down to the unsteady nonlinear IBL one:

$$\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} = 0 \tag{4}$$

$$\frac{\partial u}{\partial T} + u\frac{\partial u}{\partial X} + v\frac{\partial u}{\partial Y} = -\frac{\partial p}{\partial X} + \frac{\partial^2 u}{\partial Y^2} \tag{5}$$

$$u = v = 0 \quad \text{at} \quad Y = 0 \quad \text{(no slip)} \tag{6}$$

$$u \sim Y + A(X,T) \quad \text{as} \quad Y \to \infty \quad \text{(unknown displacement)} \tag{7}$$

$$p(X,T) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\partial A}{\partial \xi}(\xi,T)\frac{\mathrm{d}\xi}{(X-\xi)} \quad \text{(interaction law)} \tag{8}$$

Here Eq. (8) applies for subsonic two-dimensional flow. In the supersonic counterpart, Eq. (8) is replaced by Ackeret's law $p = -\partial A/\partial X$ (see also a note in Sec. IV on compressibility). There are in addition finite *Re* versions of much interest.

The two main alternatives to the nonlinear triple-deck version [Eqs. (4–8)] are the Navier-Stokes equations and IBL (and related) versions, at finite *Re*. Both of them require numerical treatments. In general, the former tends to be hindered more by grid-resolution difficulties, among others. IBL and similar methods, which are more zonal treatments involving sensible interpretation of Eqs. (4–8) at finite *Re*, have been developed only fairly recently for unsteady flows.[18-20] These have connections with interesting methods based on the parabolized Navier-Stokes equations which have also been developed recently.

Some properties of the viscous-inviscid unsteady nonlinear system [Eqs. (4–8)] and related problems are well known: see, e.g., Refs. 3 and 17. The linearized version, linearized about the original steady flow $u = Y$, $v = p = A = 0$, compares well with Orr-Sommerfeld results. It gives the neutral scaled frequency $\Omega = \Omega_n \approx 2.30$, where $\partial/\partial T \to -i\Omega$. A supercritical bifurcation is present for $\Omega > \Omega_n$, which is equivalent to $x_0 > x_{0n}$ or a displacement-thickness Reynolds number $R_\delta > R_{\delta n}$, in view of the λ factor. Computations of the nonlinear traveling states for a range of values of Ω greater than Ω_n have been presented,[29] since which more have been done. These are connected at increasing frequency with Sec. III in particular and with the reversed-flow singularities discussed in Ref. 21. Again, there are applications to linear receptivity[22,23] and nonlinear receptivity.[24] Of more concern, however, are numerical time-marching solutions of Eqs. (4–8) for the general initial-value problem. Relatively early results are presented in Refs. 25 and 26, whereas more recent accurate computations are given in Ref. 20.

Most recent interest is in the ultimate behavior of the nonlinear time-marching regime where, for finite amplitudes, the unsteady IBL system [Eqs. (4–8)] applies in full. The principal finding seems to be that localized finite-time breakups can occur[1,2]; see Fig. 1. These breakups take the form

$$\begin{aligned} X - X_s &= c(T - T_s) + (T_s - T)^N\xi \\ \frac{\partial p}{\partial X} &\sim (T_s - T)^{-1}p_1'(\xi), \quad u \to u_0(Y) \end{aligned} \tag{9}$$

near the breakup position X_s and time T_s. Here the local velocity profile u_0 is smooth, with $u_0 = c$ at the inflection point, the local coordinate ξ is of $\mathcal{O}(1)$, and the phase speed c is of $\mathcal{O}(1)$. It is found that the power $N = 3/2$, $5/4$, $7/6$, $9/8$, ..., 1, for single valuedness. In moderate breakups, $N > 1$. Then in effect an inviscid Burgers equation for the pressure function $p_1(\xi)$ describes the local terminal behavior

$$p_1 p_1' = b_1(p_1 - 3\xi p_1') \tag{10}$$

in scaled terms, from substitution into Eqs. (4–8) and integration with respect to Y. Equation (10) is for the most likely case of $N = 3/2$ and holds provided an integral constraint [Eq. (14)] on u_0 is satisfied. This yields the appropriate smooth solution

$$\xi = -b_2 p_1 - b_3 p_1^3 \tag{11}$$

implicitly, where b_2 and b_3 are constants having the same sign. Equation (11) shows that the $p_1(\xi)$ solution is single valued, as required, and monotonic in ξ. Furthermore, $|p_1| \propto |\xi|^{1/3}$ at large $|\xi|$. That asymptote matches with the flow solution further away from the breakup station $X = X_s$ and gives specifically the behavior

$$p - p_0 \propto |X - X_s|^{1/3} \quad \text{as } X \to X_s \pm \tag{12}$$

where $p_0 \equiv p(X_s)$ is a constant. Hence a singularity in the pressure gradient is predicted at the breakup time $T = T_s$. For severe breakups where $N = 1$, on the other hand, the local governing equations become

$$u = \frac{\partial \psi}{\partial y}, \quad (\xi + u)\frac{\partial u}{\partial \xi} - \frac{\partial \psi}{\partial \xi}\frac{\partial u}{\partial y} = -p_1'(\xi) \tag{13}$$

for $(u,\psi)(\xi,Y)$, $p_1(\xi)$, in normalized form. Here c is set to zero without loss of generality. Solutions are given in Ref. 2. Both sorts of breakup provoke increasingly large wall-shear responses in the local motion. Various previous unsteady IBL computations appear to support qualitatively the singular description in Eqs. (9–12). There is also fair agreement with the Navier-Stokes computations of Ref. 27, whereas experimental comparisons are described at the end of this section. Moreover, the breakups [Eqs. (9–13)] apply to most of the unsteady interactive flows known to date.

Detailed quantitative comparisons between computations of Eqs. (4–8) and the breakup theory of Eqs. (9–12), for $N = 3/2$, are made by Peridier et al.[20] showing very good agreement; see Fig. 1. In that figure, the upper graph illustrates the progression to the nonlinear breakup singularity mainly in terms of the total pressure gradient p' or normal velocity v; cf. the numerical results.[20] Table 1 compares (at various *Re* values) the values of the power B implied by the numerical results with the value $B = -¼$ implied by the theory. The agreement is felt to be encouraging. The power here occurs in the behavior $\tau_w \propto (T_s - T)^B$ of the scaled skin friction. The lower graph in the figure concerns the local vortex roll up occurring subsequently, in shorter time and length scales, as discussed next.

Following this breakup, new physical effects come into play locally associated more with the unsteady Euler equations. An appropriate computational approach in principle then is in Refs. 18 and 19. More specifically, normal pressure gradients become significant on shorter length scales. This new faster stage is discussed by Hoyle et al.,[28] where it is shown that an extended *KdV* equation holds for the pressure, subject to matching, at large negative scaled times, with the breakup [Eqs. (9–12)]. Beyond that, another new stage of still faster time scales is encountered as a strong local vortex formation takes place. This is felt to be associated closely with the initiation and eruption of a hairpin vortex from the viscous layer. Intuition would suggest also that this breakup process, when repeated again and again, may well be connected with the occurrence of intermittency in practice. Smith and Bowles[11] make comparisons between the breakup criterion[2] that arises from Eqs. (9–12), namely,

$$\int_0^\infty [u_0(Y) - c]^{-2}\,\mathrm{d}Y = 0 \tag{14}$$

Table 1 Values of the power B at various *Re* values

Re	B (from computations) 20	B (from theory) 2
10^8	-0.252 ± 0.016	-0.25
10^7	-0.253 ± 0.035	-0.25
10^6	-0.263 ± 0.022	-0.25
10^5	-0.234 ± 0.032	-0.25

and the experimental measurements of Nishioka et al.[14] concerning the first spike in transition. The agreement found is relatively close.

III. High-Frequency Disturbances and Euler-Scale Interactions (Theory 2)

An alternative to Eqs. (9–12) for the IBL system [Eqs. (4–8)] concerns what happens downstream, either as the input dimensional frequency is increased or as the station x_0 is increased.[3,29] In such cases, $\Omega \gg 1$ and an initially small disturbance passes through at least two stages (stage 1 and stage 2) downstream in the two-dimensional case. These stages are conveniently defined by their representative pressure levels:

$$p = \mathcal{O}(\Omega^{1/2}) \quad \text{(stage 1)} \tag{15}$$

and

$$p = \mathcal{O}(\Omega) \quad \text{(stage 2)} \tag{16}$$

During stage 1, the unsteady pressure response turns out to be governed by a generalized cubic Schrödinger equation, from analysis[26] of Eqs. (4–8). The principal features are that the disturbance broadens spatially in an exponentially fast form, and the maximum amplitude also grows exponentially with the amplitude solution taking on an elliptical shape. In stage 2, further downstream, the system [Eqs. (4–8)] becomes fully nonlinear but viscous forces appear at first sight to become negligible. As a result the unsteady motion there is governed by the Benjamin-Ono[30,31] equation

$$A_T + AA_X = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\partial^2 A}{\partial \xi^2} \frac{d\xi}{(X-\xi)} \tag{17}$$

for $A(X,T)$. This matches with the earlier stage 1. It can admit nonlinear traveling waves and solitary waves as solutions, and indeed recent comparisons[12] with carefully controlled, effectively two-dimensional, experiments show very good agreement with this high-frequency theory. The flow structure is subject to the existence of a viscous unsteady sublayer, however, close to the surface, and that sublayer can itself break down in a singular form to provide eruptions of vorticity into the outer inviscid zone. Again, the recent study[21] of reversed-flow singularities is relevant here, along with the work in Eq. (9) and those following. The other major features so far in the development of this sizable high-frequency disturbance are as follows:

1) The amplitude of the nonlinear disturbance is much greater than that of the basic steady flow, although we should recall that this is for the flow nearest the surface.

2) The mean shear stress at the surface is increased by an order of magnitude above its original steady laminar value (Fig. 2).

3) There are large oscillations of the unsteady shear stress about that mean.

4) Secondary instability of the inflection-point kind may well also be present. Features 1–4 continue to hold in the subsequent development of the disturbance even further downstream.

The next stage further downstream finds the disturbance stronger still, and the entire mean-flow profile is altered.[3,4] Formally, the larger disturbance amplitude and increased frequency parameter Ω ($\rightarrow Re^{1/4}$) make the dominant spatial scales of the triple-deck structure compress, so that now typically x, y, and t are all fast, of $\mathcal{O}(Re^{1/2})$. Simultaneously, the unscaled velocities $\bar{u}$ and $\bar{v}$ and the pressure $\bar{p}$ rise to $\mathcal{O}(1)$. Hence, the current structure is controlled predominantly by the Euler equations

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{18}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{p}}{\partial \bar{x}} \tag{19}$$

$$\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{\partial \bar{p}}{\partial \bar{y}} \tag{20}$$

and appropriate initial and boundary conditions. Like the earlier stage, the validity of Eqs. (18–20) is subject to the behavior of the nonlinear viscous unsteady sublayer, now of $\mathcal{O}(Re^{-3/4})$ thickness, adjoining the surface. That sublayer can act as a source of vorticity bursts [see finite-time breakup in Eq. (9) and those following and Ref. 20] and strong local vortex formation. The formations occur possibly in a random fashion and can rejuvenate the nonlinear disturbance as it continues to move downstream. This combined nonlinear structure (Fig. 3), featuring both larger-scale properties governed mainly by the Euler equations and smaller-scale properties due to viscous action and bursting, has some promising connections with full computations and/or modeling and/or experiments on transitional or turbulent boundary layers[8,27,32,33] especially if the extension to three dimensions is allowed for as in Secs. IV and V. Again, we may also regard a "fast" transition (see Introduction) as one containing sufficiently high amplitudes [$\mathcal{O}(1)$] and frequency ranges that it bypasses the earlier stages 1 and 2 and enters the fully nonlinear Euler stage [Eqs. (18–20)] directly. Similar views of bypass transitions apply also in the strongly nonlinear theories of Secs. II and IV.

Much more needs to be known about this stage, e.g., from various computational studies, to increase our flow-structural understanding. The major extra element, however, to be acknowledged is three dimensionality, which is considered specially in Sec. IV.

IV. Other Three-Dimensional Nonlinear Features and Vortex/Wave Interactions (Theory 3)

The aim now is to extend the preceding Re-large theory to allow for three-dimensional effects, as experimental findings tell us we should in general. Some of the preceding theory and conclusions can go straight through, in the various stages, but much more volatile events can take place even at relatively low amplitudes, in three dimensions. These depend even more

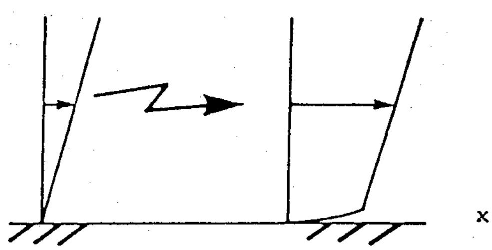

Fig. 2 Predicted nonlinear effect[3,4] of increased input frequency Ω, or downstream travel, on the mean flow. The input linear profile a develops downstream into the strongly attached mean profile b.

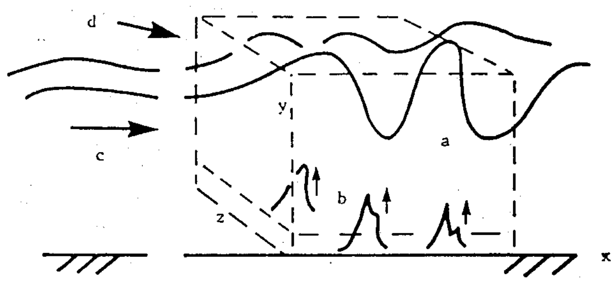

Fig. 3 Structure[3,4] of the nonlinear Euler stage a, in two or three dimensions leading to distributed sublayer bursting b. The stage results from gradual transition c or bypass transition d.

on the input spectrum and amplitudes, and there are many transition paths that can be triggered off, some of which are covered next.

Once again we may start with the triple-deck or IBL version (theory 1) since it allows for linear and nonlinear TS disturbances. The IBL in three dimensions has

$$\operatorname{div} \boldsymbol{u} = 0, \quad (\partial_T + \boldsymbol{u} \cdot \nabla)(u,w) = -(p_X, p_Z) + \partial_Y^2 (u,w) \tag{21}$$

in X,Y,Z,T coordinates, with

$$p(X,Z,T) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \times \frac{\partial^2 A}{\partial \xi^2}(\xi,\eta,T) \frac{\mathrm{d}\xi \cdot \mathrm{d}\eta}{[(X-\xi)^2 + (Z-\eta)^2]^{1/2}} \tag{22}$$

$$u \sim Y + A(X,Z,T), \quad w \to 0 \text{ as } Y \to \infty \tag{23}$$

and no-slip at $Y = 0$, as generalized from Eqs. (1–3) and Eqs. (4–8), in the incompressible regime. Computations of Eqs. (21–23) are given in Ref. 34, whereas the compressible regime is studied in Ref. 35, among others.

High-frequency (theory 2) analysis in three dimensions then shows that the "resonant-triad" mechanism can occur.[36-38] Comparisons[37] between this theory and the main experiments[39] are encouraging (see Fig. 4), yielding both small- and large-scale agreement in quantitative terms. Moreover, the three-dimensional nonlinear interaction occurs at pressure levels p of $\mathcal{O}(1)$, much less than in the corresponding two-dimensional version summarized in Eqs. (15) and (16). To maintain this relevance to experimental findings and go further into transition requires theoretical investigations of higher pressure levels, for this particular path. Some such investigations have been started by "this" author and co-workers and the current state is summarized briefly in Refs. 9 and 40 and elsewhere, e.g., Ref. 38 (and work by P. A. Stewart). The three-dimensional interactions tend to lead to an Euler stage eventually, as in Sec. III, but it is three dimensional and encountered much sooner than in the two-dimensional case. Extensions to nonlinear initial-value problems connected with the evolution of nonlinear spot disturbances are also being investigated.[15,16]

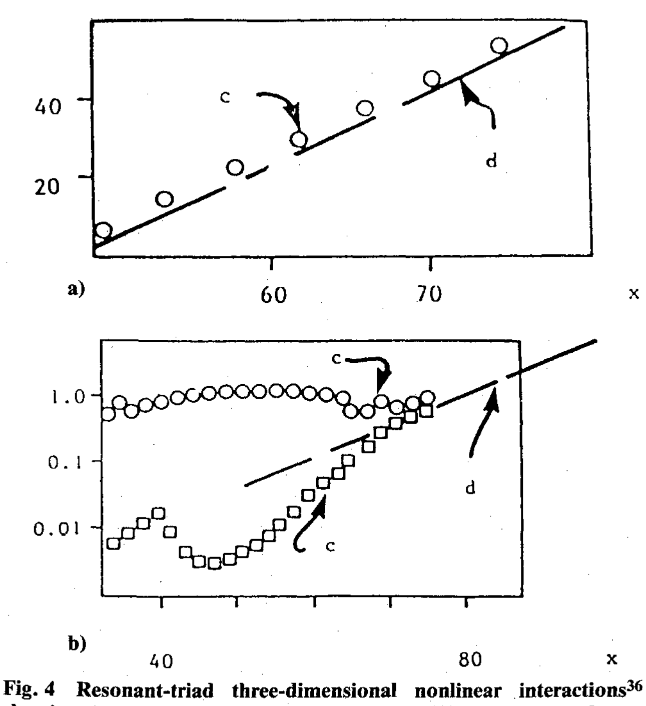

Fig. 4 Resonant-triad three-dimensional nonlinear interactions[36] showing a) phase (radians) and b) amplitudes (%) vs distance x downstream, in centimeters, and comparing experiments[38] vs downstream-flow theory.[37]

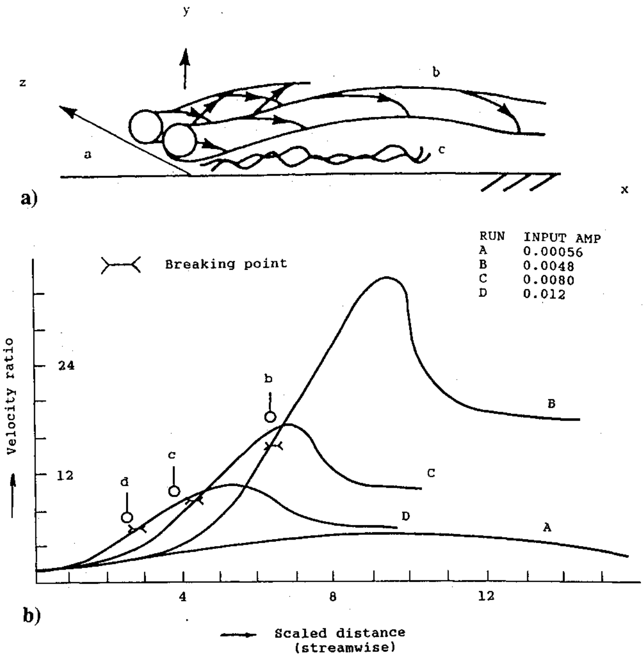

Fig. 5 a) Vortex-wave three-dimensional interactions in which the input field (at a) is converted into a mean vortex flow (b) coupled nonlinearly with accompanying TS or Rayleigh waves (c); see Refs. 5, 6, 9 and 41. b) Comparisons[10] for the dangerous amplitude-dependent three-dimensional paths following warped two-dimensional input upstream. Theoretical[10] blow-up locations b, c, d are compared with the "breaking point" locations found experimentally[13] in the runs B, C, D.

Next, for the general time-marching of the IBL equations (theory 1 again), finite-amplitude disturbances governed by Eqs. (21–23) can lead to a three-dimensional extension of the finite-time breakups described in Sec. II [Eqs. (9–13) and Fig. 1]. This is of much current interest, e.g., with regard to initiation of hairpin vortices; see Ref. 28.

In addition, however, vortex/wave interactions (theory 3) can arise at comparatively small amplitudes; see Fig. 5. There are several such nonlinear interactions possible, depending again on the disturbance environment, and much research is now in progress on these, stemming from Ref. 5. One case we wish to highlight, for instance, leads to the following vortex/TS-interaction system (Ref. 6, see also Refs. 10 and 38), in suitably scaled terms:

$$\operatorname{div} \boldsymbol{u} = 0 \tag{24}$$

$$(\partial_t + \boldsymbol{u} \cdot \nabla)(\bar{u},\bar{w}) = -(\bar{p}',0) + \partial_{\bar{y}}^2 (\bar{u},\bar{w}) \tag{25}$$

in $x,\bar{y},\bar{z},t$ coordinates, subject to

$$\bar{u} = \bar{v} = 0 \quad \text{at } \bar{y} = 0 \tag{26}$$

$$\bar{w} = -\bar{\lambda}^{-2}\partial_z [|P|^2 + \alpha^{-2}|P_z|^2] \quad \text{at } \bar{y} = 0 \tag{27}$$

$$\bar{u} \to u_e(x), \quad \bar{w} \to 0, \quad \text{as } \bar{y} \to \infty \tag{28}$$

for the three-dimensional vortex motion, i.e., the unknown mean flow throughout the boundary layer, coupled with

$$P_{zz} - F\bar{\lambda}^{-1}\bar{\lambda}_z P_z - \alpha^2 P = GA \tag{29}$$

for the unknown TS pressure. Here $\boldsymbol{u} = (\bar{u},\bar{v},\bar{w})$, $\bar{p}' = -u_e u_e'$ is the external pressure gradient, the x scale is that of the airfoil chord, i.e., $\mathcal{O}(1)$, $\bar{\lambda} = \bar{u}_{\bar{y}}$ at $\bar{y} = 0$ is the unknown mean skin friction, whereas F and G are related to the Airy function and α is the unknown real wave number. In addition, there is a P–A relation analogous with that in Eq. (22). This short-scale/long-scale interaction stems from one derived in Ref. 9 for the IBL case [Eq. (21–23)] in fact, although Eqs. (24–29) are for much reduced TS amplitudes, and has the entire mean-flow profile altered from its original steady state. Computational space-marching properties are described in Ref. 41, further to the investigations in Ref. 9 of the ultimate behavior of the nonlinear solutions downstream. The latter indicate possible agreement with experiments on lambda-vortex or hairpin-vortex formation, as a type of three-dimensional separation singularity,[9] following which inner-outer interaction and breakup are anticipated locally as in Sec. II. The three-dimensional separation singularity gives the characteristic form

$$\delta \sim (x_1 - x)^{-m}, \quad \text{as } x \to x_1 - \tag{30}$$

for the boundary-layer displacement, with $m = \frac{1}{2}$ or $\frac{1}{3}$ as the major possibilities, from Ref. 9 and from further investigations.[42,43] We should mention that three-dimensional attachment singularities are also possible.[9] Related research on compressibility effects (see also Ref. 35) and on pipe flows has also been done.[43,44] Again, vortex/Rayleigh-wave interactions are also set out in Ref. 41, following which Brown et al.[45] and Smith et al.[46] study some intriguing new short-scale effects.

Another vortex/wave interaction we should highlight is Stewart and Smith's,[10] as it appears to provide the first explanation of the classic transition path of Klebanoff and Tidstrom.[13] It stems from a high-frequency analysis of Eqs. (21–23) and as such it brings together all three theories in a sense. Moreover, the pressure level is relatively quite low, of order unity, with

$$p = \exp[i(X_1 - T_1)]\,\tilde{p}\,(X_2, Z, T_2) + \text{c.c.} + \ldots \tag{31}$$

thus emphasizing the power of this short-scale/long-scale interaction; compare with Eqs. (15) and (16) and the pressure level in the preceding resonant triad. Here the multiscales present are $(X_1, X_2) = (\Omega^{1/2}, \Omega^{-1/2})X$, $(T_1, T_2) = (\Omega, 1)T$, but Z is $\mathcal{O}(1)$ as the input is almost two-dimensional. Substitution into Eqs. (21–23) yields after some working the nonlinear system

$$\tilde{p}_s - i\tilde{p}_{ZZ} = \tilde{p} - i\tilde{p}Q \tag{32}$$

$$Q_{ss} = (|\tilde{p}|^2)_{ZZ} \tag{33}$$

for $\tilde{p}, Q$. Here Q is the mean-flow correction and s is the appropriate moving coordinate (in X_2, T_2). An important point is that, whereas a purely two-dimensional evolution with zero $\partial/\partial Z$ would remain linear, the slight three dimensionality (or warping) present here forces a relatively early nonlinear response, far earlier than in Eq. (17) for example. Analytical and numerical properties are found[10] which confirm the formation of three-dimensional streets and lead to quantitative comparisons with experiments[13] as shown in Fig. 5. (Similar comparisons with Nishioka et al.'s[14] experiments are among those presented in Ref. 11.) The agreement is encouraging.

V. Further Comments

It may be speculated that certain of the above features, whether in two dimensions or three dimensions, provide the building blocks for the structure of turbulent boundary layers. This is particularly true for the three strongly nonlinear interactions of Secs. II–IV in which the mean-flow profile is affected substantially. There is in addition much encouragement from the tie up with more modeling-based work[8] and the associated experimental links in the turbulent stage. Thus, while there is undoubtedly a vast amount still unknown on the theoretical side, there is now a fair range of quite supportive agreements between theory as so far developed and experiments in both the transitional and the turbulent stages.

The Euler stage (theory 2) of Sec. III yields three features, in particular, which resemble three major properties of turbulent boundary layers in practice, namely 1) the large thickening compared with the steady laminar version, 2) the emergence of a very thin wall layer, in the mean flow, and 3) the active microscale for small eddies. The prediction[4] for 1) is a thickness

$$\mathcal{O}\,(Re^{-1/2}\ln Re) \tag{34}$$

in the two-dimensional case and for 2) a thickness

$$\mathcal{O}\,(Re^{-1}\ln Re) \tag{35}$$

from an argument involving scale cascades and renormalization. These may be compared qualitatively with experiments[47,48] and specifically with the empirical values (e.g., Ref. 49) of $\mathcal{O}(u_\tau)$, $\mathcal{O}(Re^{-1}u_\tau^{-1})$ [where $u_\tau \sim (\ln Re)^{-1}$], in turn. The disagreement for 1) is probably due to three-dimensional effects, whereas the exact agreement on 2) is clearly an encouraging feature. The Kolmogorov microscale 3) of

$$\mathcal{O}\,(Re^{-3/4}) \tag{36}$$

is also verified exactly in Ref. 4, along with the connections with slugs and puffs[50,51] in channel and pipe flows, whereas the evolution of nonlinear spot-like disturbances is currently under consideration.[15,16]

The nonlinear TS or IBL stage (theory 1) of Sec. II, which may be linked with intermittency, occurs also in the viscous sublayer induced during the Euler stage. It yields sublayer eruption and strong vortex formation locally, again integral parts of a turbulent boundary-layer structure. Indeed, the theoretical predictions of eruption and vortex formation[20,28] seem likely to be connected with the start of the hairpin vortices that are proposed[8] as major constituents of turbulent boundary layers. Again, see the theoretical-experimental comparisons in Ref. 11. The complete role of vortex/wave interactions (theory 3, Sec. IV) is less clear as yet, but potentially powerful. They may act as readily accessible precursors to 1) inner-outer interaction, through a displacement singularity, leading on then to local breakup as in Sec. II; and/or 2) a cascade of length and time scales locally, associated with the vortex/Rayleigh case. Encouraging theoretical-experimental comparisons have emerged recently[10,11] for various stages in one of the classic transition paths.

Acknowledgments

Thanks are due to S.E.R.C., U.K., to the United Technologies Independent Research Program, to Air Force Office of Scientific Research, Grant 89-0475, and A.R.O., Contract DAAL03-92-G-0040, for their support of certain parts of this research, and to many colleagues.

References

[1]Brotherton-Ratcliffe, R. V., and Smith, F. T., "Complete Breakdown of an Unsteady Interactive Boundary Layer (Over a Surface Distortion or in a Liquid Layer)," *Mathematika*, Vol. 34, 1987, pp. 86–100.

[2]Smith, F. T., "Finite-Time Breakup Can Occur in Any Unsteady Interacting Boundary Layer," *Mathematika*, Vol. 35, 1988, pp. 256–273.

[3]Smith, F. T., and Burggraf, O. R., "On the Development of Large-Sized Short-Scaled Disturbances in Boundary Layers," *Proceedings of the Royal Society*," Vol. A399, 1985, pp. 25–255.

[4]Smith, F. T., Doorly, D. J., and Rothmayer, A. P., "On Displacement-Thickness, Wall-Layer and Mid-Flow Scales in Turbulent Boundary Layers, and Slugs of Vorticity in Pipes and Channels," *Proceedings of the Royal Society*, Vol. A428, 1990, pp. 255–281.

[5]Hall, P., and Smith, F. T., "The Nonlinear Interaction of Tollmien-Schlichting Waves and Taylor-Gortler Vortices in Curved Channel Flows," *Proceedings of the Royal Society*, Vol. A417, 1988, pp. 255–282.

[6]Hall, P., and Smith, F. T., "Nonlinear Tollmien-Schlichting/Vor-

tex Interaction in Boundary Layers," *European Journal of Mechanics*, Vol. B8, 1989, pp. 179–205; also Institute for Computer Applications in Science and Engineering, Rept. 88-46, 1981.

[7]Walker, J. D. A., "Wall-Layer Eruptions in Turbulent Flows," *Proceedings, Second International Union of Theoretical and Applied Mechanics Symposium on Structure of Turbulence and Drag Reduction*, edited by A. Gyr, Springer-Verlag, New York, 1990, pp. 109–117.

[8]Smith, C. R., Walker, J. D. A., Haidari, A. H., and Sobrun, U., "On the Dynamics of Near-Wall Turbulence," *Philosophical Transactions of the Royal Society*, Vol. A336, 1991, pp. 131–175.

[9]Smith, F. T., and Walton, A. G., "Nonlinear Interaction of Near-Planar TS Waves and Longitudinal Vortices in Boundary-Layer Transition," *Mathematika*, Vol. 36, 1989, pp. 262–289.

[10]Stewart, P. A., and Smith, F. T., "Three-Dimensional Nonlinear Blow-Up from a Nearly Planar Initial Disturbance in Boundary-Layer Transition: Theory and Experimental Comparisons," *Journal of Fluid Mechanics*, Vol. 244, 1992, pp. 79–100.

[11]Smith, F. T., and Bowles, R. I., "Transition Theory and Experimental Comparisons on (I) Amplification into Streets and (II) a Strongly Nonlinear Breakup Criterion," *Proceedings of the Royal Society*, Vol. A439, 1992, pp. 163–175.

[12]Kachanov, Y. S., Ryzhov, O. S., and Smith, F. T., "Formation of Solitons in Transitional Boundary Layers: Theory and Experiment," *Journal of Fluid Mechanics*, Vol. 251, 1993, pp. 273–297.

[13]Klebanoff, P. S., and Tidstrom, K. D., "The Evolution of Amplified Waves Leading to Transition in a Boundary Layer With Zero Pressure Gradient," NASA TN D-195, 1959; also Stuart, J. T., *Laminar Boundary Layers*, edited by L. Rosenhead, Oxford Univ. Press, England, UK, 1963, Chap. IX.

[14]Nishioka, M., Asai, M., and Iida, S., "An Experimental Investigation of the Secondary Instability," *Laminar-Turbulent Transition*, International Union of Theoretical and Applied Mechanics Meeting, (Stuttgart), Springer-Verlag, Berlin, 1979.

[15]Smith, F. T., "On Nonlinear Effects Near the Wing-Tips of an Evolving Boundary-Layer Spot," *Philosophical Transactions of the Royal Society*, Vol. A340, 1992, pp. 131–165.

[16]Smith, F. T., and Doorly, D. J., "Initial-Value Problems for Spot Disturbances in Incompressible or Compressible Boundary Layers," *Journal of Engineering Mathematics*, Vol. 26, 1992, pp. 87–106.

[17]Smith, F. T., "Nonlinear Stability of Boundary Layers for Disturbances of Various Sizes," *Proceedings of the Royal Society*, Vol. A368, 1979, pp. 573–589.

[18]Smith, F. T., Papageorgiou, D. T., and Elliott, J. W., "An Alternative Approach to Linear and Nonlinear Stability Calculations at Finite Reynolds Numbers," *Journal of Fluid Mechanics*, Vol. 146, 1984, pp. 313–330.

[19]Smith, F. T., "Composite, Navier-Stokes and Euler Unsteady-Flow Computations in Boundary Layers," United Technologies Research Center, Rept. UTRC91-2, Hartford, CT, July 1991.

[20]Peridier, V. J., Smith, F. T., and Walker, J. D. A., "Vortex-Induced Boundary-Layer Separation" (Parts I, II), *Journal of Fluid Mechanics*, Vol. 232, 1991, pp. 99–131 and pp. 133–165; also Peridier, J., Ph.D. Thesis, Lehigh Univ. Bethlehem, PA, 1989.

[21]Smith, F. T., "A Reversed-flow Singularity In Interacting Boundary Layers," *Proceedings of the Royal Society*, Vol. A420, 1988, pp. 21–52.

[22]Goldstein, M. E., "Scattering of Acoustic Waves into Tollmien-Schlichting Waves by Small Streamwise Variations in Surface Geometry," *Journal of Fluid Mechanics*, Vol. 154, 1985, pp. 509–530.

[23]Goldstein, M. E., and Hultgren, L. S., "Boundary-Layer Receptivity to Long-Wave Freestream Disturbances," *Annual Reviews of Fluid Mechanics*, Vol. 21, 1989, pp. 137–166.

[24]Smith, F. T., "Breakup in Unsteady Separation," *Transactions of the ASME Conference on Unsteady Separation*, Cincinnati, OH, 1987.

[25]Duck, P. W., "Laminar Flow Over Unsteady Humps: The Formation of Waves," *Journal of Fluid Mechanics*, Vol. 160, 1985, pp. 465–498.

[26]Smith, F. T., "Two-Dimensional Disturbance Travel, Growth and Spreading in Boundary Layers," United Technologies Research Center, Rept. UTRC85-36, Hartford, CT, 1985; see also AIAA Paper 84-1582, 1984.

[27]Fasel, H., "Numerical Investigation of Two-Dimensional Development in Boundary-Layer Transition," *Proceedings, Symposium on Turbulent & Chaotic Phenonema in Fluids*, edited by T. Tatsumi, Elsevier, New York, 1984, pp. 94–103.

[28]Hoyle, J. M., Smith, F. T., and Walker, J. D. A., "On Sublayer Eruption and Vortex Formation," *Computational Physics Communications*, Vol. 65, 1991, pp. 151–157; also Hoyle, J.M., Ph.D. Thesis, Mathematics Dept., Univ. College London, London Univ., London, 1992.

[29]Conlisk, A. T., Burggraf, O. R., and Smith, F. T., "Nonlinear Neutral Modes in the Blasius Boundary Layer," *Transactions of the ASME, Forum on Unsteady Separation*, Cincinnati, OH, June 1987.

[30]Benjamin, T. B., "Internal Waves of Permanent Form in Fluids of Great Depth," *Journal of Fluid Mechanics*, Vol. 29, 1967, pp. 559–592.

[31]Ono, H., "Algebraic Solitary Waves in Stratified Fluids," *Journal of the Physical Society Japan*, Vol. 39, 1975, pp. 1082–1091.

[32]Walker, J. D. A., Abbott, D. E., Sharnhorst, R. K., and Weigan, G. G., "Wall-Layer Model for Velocity Profile in Turbulent Flows," *AIAA Journal*, Vol. 27, No. 2, 1989, pp. 140–149.

[33]Orszag, S. A., and Patera, A. T., "Secondary Instability of Wall-Bounded Shear Flows," *Journal of Fluid Mechanics*, Vol. 128, 1983, pp. 347–386.

[34]Smith, F. T., "Steady and Unsteady 3-D Interactive Boundary Layers," *Computers and Fluids*, Vol. 20, 1991, pp. 243–268.

[35]Smith, F. T., "On the First-Mode Instability in Subsonic, Supersonic, or Hypersonic Boundary Layers," *Journal of Fluid Mechanics*, Vol. 198, 1989, pp. 127–153.

[36]Craik, A. D. D., "Non-Linear Resonant Instability in Boundary Layers," *Journal of Fluid Mechanics*, Vol. 50, 1971, pp. 393–413.

[37]Smith, F. T., and Stewart, P. A., "The Resonant-Triad Nonlinear Interaction in Boundary-Layer Transition," United Techologies Research Center, Rept. UTRC86-26, Hartford, CT, 1986; also *Journal of Fluid Mechanics*, Vol. 179, 1987, pp. 227–252.

[38]Smith, F. T., Stewart, P. A., and Bowles, R. G. A., "On the Nonlinear Growth of Single Three-Dimensional Disturbances in Boundary Layers," *Mathematika*, 1994 (to be published).

[39]Kachanov, Y. S., and Levchenko, V. Y., "The Resonant Interaction of Disturbances at Laminar-Turbulent Transition in a Boundary Layer," *Journal of Fluid Mechanics*, Vol. 138, 1984, pp. 209–248.

[40]Smith, F. T., "On Transition to Turbulence in Boundary Layers," *European Conference on Turbulence*, Springer-Verlag, Berlin, 1987, pp. 1–10; also United Technologies Research Center, Rept. UTRC86-10, Hartford, CT, 1986.

[41]Hall, P., and Smith, F. T., "On Strongly Nonlinear Vortex-Wave Interactions in Boundary-Layer Transition," *Journal of Fluid Mechanics*, Vol. 227, 1991, pp. 641–666; also Institute for Computer Applications in Science and Engineering, Rept. 89-92, 1989.

[42]Walton, A. G., and Smith, F. T., "Properties of Strongly Nonlinear Vortex/Tollmien-Schlichting-Wave Interactions," *Journal of Fluid Mechanics*, Vol. 244, 1992, pp. 649–676.

[43]Walton, A. G., "Theory and Computation of Three-Dimensional Nonlinear Effects in Pipe Flow Transition," Ph.D. Thesis, Mathematics Dept., Univ. of London, London, 1991.

[44]Blackaby, N. D., "On Viscous, Inviscid and Centrigal Instability Mechanisms in Compressible Boundary Layers, Including Nonlinear Vortex/Wave Interaction and the Effects of Large Mach Number on Transitions," Ph.D. Thesis, Mathematics Dept., Univ. of London, London, 1991.

[45]Brown, P. G., Brown, S. N., Smith, F. T., and Timoshin, S. N., "On the Starting Process of Strongly Nonlinear Vortex/Rayleigh-Wave Interactions," *Mathematika*, Vol. 40, 1993, pp. 7–29.

[46]Smith, F. T., Brown, P. G., and Brown, S. N., "Initiation of Three-Dimensional Nonlinear Transition Paths from an Inflexional Profile," *European Journal of Mechanics* (to be published).

[47]Gad-el-Hak, M., Blackwelder, R. E., and Riley, J. J., "On the Growth of Turbulent Regions in Laminar Boundary Layers," *Journal of Fluid Mechanics*, Vol. 110, 1981, pp. 73–96.

[48]Riley, J. J., and Gad-el-Hak, M., "The Dynamics of Turbulent Spots," *Frontiers in Fluid Mechanics*, edited by S. H. Davis and J. L. Lumley, Springer-Verlag, New York, 1985, pp. 123–155.

[49]Cebeci, T., and Smith, A. M. O., *Turbulent Shear Flows*, Academic Press, New York, 1974.

[50]Wygnanski, I. J., and Champagne, F. H., "On Transition in a Pipe. Part 1. The Origin of Puffs and Slugs and the Flow in a Turbulent Slug," *Journal of Fluid Mechanics*, Vol. 59, 1973, pp. 281–336.

[51]Wygnanski, I. J., Sokolov, M., and Friedman, D., "On Transition in a Pipe. Part 2. The Equilibrium Puff," *Journal of Fluid Mechanics*, Vol. 69, 1975, pp. 283–304.